Finite Math - J-term 2017

Lecture Notes - 1/6/2017

Homework

- Section 3.2 - 9, 11, 13, 15, 17, 23, 33, 35, 43, 45, 47, 49, 57, 61, 66, 72, 73, 76, 77, 58, 87, 89

Section 3.2 - Compound and Continuous Compound Interest

Compound Interest. In the case of simple interest, the interest is computed exactly once: at the end. Typically, however, interest is usually compounded something like monthly or quarterly.

Example 1. *Suppose* $\$5,000$ *is invested at* 12%*, compounded quarterly. How much is the investment worth after* 1 *year?*

Solution. *We find the future value at the end of the first quarter:*

$$A_1 = \$5,000\left(1 + 0.12\left(\frac{1}{4}\right)\right) = \$5,150.$$

This amount is carried into the second quarter and interest is computed again over the quarter:

$$A_2 = \$5,150\left(1 + 0.12\left(\frac{1}{4}\right)\right) = \$5,304.50.$$

We do this twice more to find a value at the end of the fourth quarter:

$$A_3 = \$5,304.50\left(1 + 0.12\left(\frac{1}{4}\right)\right) = \$5,463.635.$$

$$A_4 = \$5,463.635\left(1 + 0.12\left(\frac{1}{4}\right)\right) = \$5,627.54.$$

If we generalize this process, we end up with the following result

Definition 1 (Compound Interest)**.**

$$A = P(1+i)^n, \text{ where } i = \frac{r}{m}$$

The variables in this equation are

- $A =$ *future value after* n *compounding periods*

- $P =$ *principal*
- $r =$ *annual nominal rate*
- $m =$ *number of compounding periods per year*
- $i =$ *rate per compounding period*
- $n =$ *total number of compounding periods*

Alternately, one can reinterpret this formula as a function of time as

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

where A, P, r, and m have the same meanings as above and t is the time in years.

Example 2. *If* $\$1,000$ *is invested at* 6% *interest compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, what is the value of the investment after* 8 *years? Round answers to the nearest cent.*

Solution. *In this example, the quantities that will be changing are m and n (and thus also i). The fixed quantities are the principal $P = \$1,000$ and the annual nominal rate $r = 0.06$.*

(a) *Annually compounded means $m = 1$. Since we are going for 8 years, this means there will be $n = 8(1) = 8$ compounding periods. We also get $i = \dfrac{0.06}{1} = 0.06$, so the future value will be*

$$A = \$1,000(1 + 0.06)^8 = \$1,593.85.$$

(b) *Semiannually compounded means $m = 2$. Since we are going for 8 years, this means there will be $n = 8(2) = 16$ compounding periods. We also get $i = \dfrac{0.06}{2} = 0.03$, so the future value will be*

$$A = \$1,000(1 + 0.03)^{16} = \$1,604.71.$$

(c) *Quarterly compounded means $m = 4$. Since we are going for 8 years, this means there will be $n = 8(4) = 32$ compounding periods. We also get $i = \dfrac{0.06}{4} = 0.015$, so the future value will be*

$$A = \$1,000(1 + 0.015)^{32} = \$1,610.32.$$

(d) *Monthly compounded means* $m = 12$. *Since we are going for* 8 *years, this means there will be* $n = 8(12) = 96$ *compounding periods. We also get* $i = \dfrac{0.06}{12} = 0.005$, *so the future value will be*

$$A = \$1,000(1 + 0.005)^{96} = \$1,614.14.$$

Example 3. *If* \$2,000 *is invested at* 7% *compounded (a) annually, (b) quarterly, (c) monthly, what is the amount after* 5 *years? How much interest is accrued in each case? Round answers to the nearest cent.*

Solution.

(a) \$2805.10 *with* \$805.10 *in interest.*

(b) \$2829.56 *with* \$829.56 *in interest.*

(c) \$2835.25 *with* \$835.25 *in interest.*

Continuous Compound Interest. Consider again the formulation of compound interest given by

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

We can do the following manipulation to this expression

$$\begin{aligned} A &= P\left(1 + \frac{r}{m}\right)^{mt} \\ &= P\left(1 + \frac{r}{m}\right)^{mt\cdot\frac{r}{r}} \\ &= P\left(1 + \frac{r}{m}\right)^{\left(\frac{m}{r}\right)rt} \\ &= P\left(1 + \frac{1}{x}\right)^{xrt} \qquad \left(x = \frac{m}{r}\right) \\ &= P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt} \end{aligned}$$

Now, if we let the number of compounding periods per year m get very very large, then x also gets very large, and we see that the future value becomes

$$A = Pe^{rt}.$$

Definition 2 (Continuous Compound Interest)**.** *Principal* P *invested at an annual nominal rate* r *will have future value*

$$A = Pe^{rt}$$

after time t *(in years).*

Compounding interest continuously gives the absolute largest amount of interest that can be accumulated in the time period t.

Example 4. *If* $\$1,000$ *is invested at* 6% *interest compounded continuously, what is the value of the investment after* 8 *years? Round answers to the nearest cent.*

Solution. *The principal is* $\$1,000$ *and the interest rate is* $r = 0.06$ *with a time of* $t = 8$ *years, so the future value in this case is*

$$A = \$1,000e^{(0.06)(8)} = \$1,000e^{0.48} = \$1,616.07$$

which is we see is larger than any of the others.

Example 5. *If* $\$2,000$ *is invested at* 7% *compounded (a) daily, (b) continuously, what is the amount after* 5 *years? How much interest is accrued in each case? Round answers to the nearest cent. (Assume* 365 *days in a year.)*

Solution.

(a) $\$2838.04$ *with* $\$838.04$ *in interest.*

(b) $\$2838.14$ *with* $\$838.14$ *in interest.*

As before, we can use these compound interest models to figure out how much we should invest now to achieve a desired future value.

Example 6. *New parents are looking at a college savings account which gives* 8% *interest. If they are looking to have* $\$80,000$ *when their child is ready to go to college in* 17 *years, how much should they invest now if interest is compounded (a) semiannually, (b) continuously? Round answers to the nearest cent.*

Solution. *Here, we have* $r = 0.08$*,* $t = 17$*, and* $A = \$80,000$ *as given values. We are looking for the principal in both cases.*

(a) *If interest is compounded semiannually, then the interest per compounding period is* $i = \frac{0.08}{2} = 0.04$ *and the number of compounding periods is* $n = 2(17) = 34$*. So the formula gives us*

$$\$80,000 = P(1 + 0.04)^{34} = 3.794316P$$

and solving for P *says that the principal the parents should invest is*

$$P = \$21,084.17.$$

(b) *If interest is compounded continuously, then*

$$\$80,000 = Pe^{(0.08)(17)} = 3.896193P$$

so the principal is

$$P = \$20,582.36.$$

Example 7. *You are looking at a retirement account which pays* 2% *interest. If you are looking to have* $\$1,000,000$ *in the account by the time you retire in* 50 *years how much should you invest now if interest is compounded (a) quarterly, (b) continuously? Round answers to the nearest cent.*

Solution.

(a) $\$368,797.23$

(b) $\$367,879.44$

We can also look to see how long something will take to mature given the principal, the growth rate, and the desired future value. The power rule for logarithms comes especially in handy here: $\log_b M^p = p \log_b M$.

Example 8. *How long will it take* $\$10,000$ *to grow to* $\$25,000$ *if it is invested at* 8% *compounded quarterly?*

Solution. *Here, we have* $P = \$10,000$*,* $A = \$25,000$*,* $r = 0.08$*,* $i = \frac{0.08}{4} = 0.02$*, thus the model gives us*

$$\$25,000 = \$10,000(1 + 0.02)^n$$

and so we can solve for the number of compounding periods required.

$$\begin{aligned} \$25,000 &= \$10,000(1 + 0.02)^n \\ 2.5 &= 1.02^n \\ \ln 2.5 &= n \ln 1.02 \\ n &= \frac{\ln 2.5}{\ln 1.02} = 46.27 \end{aligned}$$

So, this means we need 47 *quarters to achieve* $\$25,000$*, or* 11 *years and* 3 *quarters.*